Ministerul Educaţiei, Cercetării şi Inovării
Societatea de Ştiinţe Matematice din România

Proba 1 de Selecţie pentru Lotul Olimpic
Neptun, 15 Aprilie 2009

Problem 1. For non-empty subsets $A, B \subseteq \mathbb{Z}$ let us define $A+B$ and $A-B$ by[1]

$$A+B=\{a+b \mid a \in A,\, b \in B\},\ A-B=\{a-b \mid a \in A,\, b \in B\}.$$

In the sequel we work with non-empty finite subsets of $\mathbf{Z}$. Prove that we can *cover* B by at most $\frac{|A+B|}{|A|}$ *translates* of $A-A$, i.e. there exists $X \subseteq \mathbf{Z}$ with $|X| \le \frac{|A+B|}{|A|}$ such that

$$B \subseteq \bigcup_{x \in X} (x+(A-A)) = X+A-A.$$

Imre Rusza[2]

Solution. Consider the sets $x+A$, where x ranges over B. They are all contained in $A+B$, each of cardinality $|A|$. Consider a maximal disjoint family of such sets (e.g. by a greedy algorithm). There are therefore at most $\frac{|A+B|}{|A|}$ such $x+A$ sets in this family, and denote by $X \subseteq B$ the set of the x's. Now let b be any element of B, so, by maximality, $b+A$ must intersect (at least) one set in the family above, i.e. there exists a set $x+A$ such that $(b+A) \cap (x+A) \neq \emptyset$, or $b \in x+A-A$, i.e. $b \in X+A-A$, hence X is a fitting set of translating values, and the problem is solved. ■

Problem 2. Consider a matrix whose entries are integers. Adding a same integer to all entries on a same row, or on a same column, is called an *operation*. It is given that, for infinitely many positive integers n, one can obtain, through a finite number of operations, a matrix having all entries divisible by n.

Prove that, through a finite number of operations, one can obtain the null matrix.

Marius Cavachi

Solution. (Călin Popescu) Let the given matrix be (a_{st}). For any rows i, j, $i \neq j$, and columns k, ℓ, $k \neq \ell$, the quantity $(a_{ik}-a_{i\ell})-(a_{jk}-a_{j\ell})$ is clearly invariated by any operation. Take now n any of the (infinitely many) warranted numbers, such that $n > \max|(a_{ik}-a_{i\ell})-(a_{jk}-a_{j\ell})|$. Since all the elements of the matrix obtained are divisible by n, it follows that $a_{ik}-a_{i\ell} = a_{jk}-a_{j\ell}$. This means that any column k differs from column 1, element by element, by a constant x_k, and similarly, any row i differs from row 1, element by element, by a constant y_i. Performing all operations with x_k on column k, and all with y_i on row i, one obtains the null matrix. ■

Problem 3. Some $n > 2$ lamps are cyclically connected: lamp 1 with lamp 2, …, lamp k with lamp $k+1$, …, lamp $n-1$ with lamp n, lamp n with lamp 1. At the beginning all lamps are off. When one pushes the switch of a lamp, that lamp and the two ones connected to it change status (from off to on, or vice-versa). Determine the number of the configurations of lamps reachable from the initial one, through some set of switches being pushed.

Dan Schwarz

Solution. (Andrei Neguţ) Let us label the lamps by integer numbers modulo n, in clockwise order. By pushing the switches of the lamps $i+1$ and $i+2$ one obtains the following effect: lamps i and $i+3$ change status, while all the others remain unchanged. By iterating this procedure k times, one obtains the following effect: lamps i and $i+3k$ change status, while all the others remain unchanged $(*)$.

When n is not a multiple of 3, we will show that all configurations are reachable from the initial one. Equivalently, we will show that from any configuration we can reach the one with all lamps off. For any two lamps i and j there exists k such that $j \equiv i+3k \pmod{n}$. By $(*)$, this means that we can turn off any pair of lamps. If a configuration has an even number of lamps on, this means that we can turn all of them off in pairs. If a configuration has an odd number of lamps on, we can push the switch on lamp 1, thus obtaining an even number of lamps on, and then turn them all off in pairs. Thus, in this case all configurations are reachable, so the desired number is 2^n.

When n is a multiple of 3, for any configuration let a_i denote the number of lamps on with labels $\equiv i \pmod 3$. One notices that any push of a switch changes the parity of the numbers (a_0, a_1, a_2). Therefore a configuration is reachable from the initial one only if (a_0, a_1, a_2) are all even or all odd. We will now prove the converse: if the numbers (a_0, a_1, a_2) of a certain configuration are all even or all odd, then that configuration is reachable. Equivalently, we will show that all the lamps of the configuration can be turned off by a sequence of moves. If (a_0, a_1, a_2) are all even, then we can use $(*)$ to turn all the lamps off in pairs. If (a_0, a_1, a_2) are all odd, we can push any switch to make them all even, and then use $(*)$ to turn all the lamps off in pairs. Therefore, the number of reachable configurations equals the number of configurations with (a_0, a_1, a_2) all even or all odd. One easily sees that this number is $2^{n-3}+2^{n-3}=2^{n-2}$. ■

Alternative Solution.[3] There are 2^n possible on/off configurations of the lamps, as well as 2^n subsets of the set of lamps, each of this set $\mathscr{P}$ inducing through switches, from the given all off configuration, one from the set $\mathscr{R}$ of possible reachable configurations. The effect of combining two sets S_1 and S_2 of switches is easily seen as being equivalent to applying the set $S_1 \Delta S_2$ of switches, since a switch acts as a *toggle*.

The surjective mapping $\varphi : \mathscr{P} \to \mathscr{R}$ induces on $\mathscr{R}$ a group structure derived from $(\mathscr{P}, \Delta)$, such that φ becomes a group homomorphism. The abelian group $\mathscr{R}$ then has all (but the neutral) its elements of order two, hence its order is some power $1 \leq k \leq n$ of 2, so $|\mathscr{R}| = 2^k$, and thus $|\mathrm{Ker}(\varphi)| = 2^{n-k}$.

Therefore, when the given all off configuration can be reached in one only way (by making no switch), it means that $\mathrm{Ker}(\varphi) = \{\emptyset\}$, hence all 2^n possible configurations are reachable, while when the given all off configuration can be reached in more than one way, then the number of these ways is 2^{n-k}, and 2^k configurations are reachable.

However, the only case when the initial all off configuration is reachable other than by making no switch at all, is when $n = 3m$, with either of the three sets of switches $(xxo)_m$, $(xox)_m$, $(oxx)_m$ (where x indicates a switch of the corresponding lamp, while o indicates no switch).

Therefore the answer must be 2^n when n is congruent with 1 or 2 modulo 3, and 2^{n-2} when n is congruent with 0 modulo 3. ■

---

[1] This is the Minkowski sum (or difference) of sets. By extension, $x \pm Y = \{x \pm y \mid y \in Y\}$.

[2] This obscure, but beautiful Lemma is buried in some Terence Tao lecture notes, although marred by innocuous mistakes in the proof. Culled and proofread by D. Schwarz.

[3] The solution is purposely written using modern algebra terminology, but is readily translated into more classical terms.